

LYTLE

Conditions
Under which Functions
May be Represented
By Infinite Series

Mathematics

A. M.

1904

UNIVERSITY OF ILLINOIS
LIBRARY

BOOK
1904

CLASS
L99

VOLUME



267
45 x 49

CONDITIONS UNDER WHICH FUNCTIONS
MAY BE REPRESENTED BY
INFINITE SERIES

BY

ERNEST BARNES LYCLE, B.S., '01

THESIS FOR THE DEGREE MASTER OF ARTS
IN THE GRADUATE SCHOOL
UNIVERSITY OF ILLINOIS

PRESENTED JUNE 1904



Digitized by the Internet Archive
in 2013

Contents.

	Page 3.
Introduction	
§ 1. When $\phi(x)$ is continuous	11.
§ 2. When $\phi(x)$ has a single point of discontinuity	14.
§ 3. When $\phi(x)$ has a finite number of points of discontinuity	19.
§ 4. When the points of discontinuity of $\phi(x)$, M , are infinite but have a single limiting point ...	24.
§ 5. When M , the points of discontinuity of $\phi(x)$, is discrete	30.
§ 6. When M is of the first species	34.
§ 7. When M is enumerable	37.
§ 8. When M is non-enumerable but discrete,	43.
§ 9. When M is perfect and in no sub- interval everywhere dense, and $\phi(x)$ is continuous in the domain M	44.
§ 10. When $M_\Omega = 0$	52.
§ 11. When $\phi(x)$ is pointwise dis- continuous with respect to every perfect set	59.

STANDARD
LIBRARY
UNIVERSITY OF
ILLINOIS

Continued

11. 100 1.2
12. 100 1.2
13. 100 1.2
14. 100 1.2
15. 100 1.2
16. 100 1.2
17. 100 1.2
18. 100 1.2
19. 100 1.2
20. 100 1.2
21. 100 1.2
22. 100 1.2
23. 100 1.2
24. 100 1.2
25. 100 1.2
26. 100 1.2
27. 100 1.2
28. 100 1.2
29. 100 1.2
30. 100 1.2
31. 100 1.2
32. 100 1.2
33. 100 1.2
34. 100 1.2
35. 100 1.2
36. 100 1.2
37. 100 1.2
38. 100 1.2
39. 100 1.2
40. 100 1.2
41. 100 1.2
42. 100 1.2
43. 100 1.2
44. 100 1.2
45. 100 1.2
46. 100 1.2
47. 100 1.2
48. 100 1.2
49. 100 1.2
50. 100 1.2
51. 100 1.2
52. 100 1.2
53. 100 1.2
54. 100 1.2
55. 100 1.2
56. 100 1.2
57. 100 1.2
58. 100 1.2
59. 100 1.2
60. 100 1.2
61. 100 1.2
62. 100 1.2
63. 100 1.2
64. 100 1.2
65. 100 1.2
66. 100 1.2
67. 100 1.2
68. 100 1.2
69. 100 1.2
70. 100 1.2
71. 100 1.2
72. 100 1.2
73. 100 1.2
74. 100 1.2
75. 100 1.2
76. 100 1.2
77. 100 1.2
78. 100 1.2
79. 100 1.2
80. 100 1.2
81. 100 1.2
82. 100 1.2
83. 100 1.2
84. 100 1.2
85. 100 1.2
86. 100 1.2
87. 100 1.2
88. 100 1.2
89. 100 1.2
90. 100 1.2
91. 100 1.2
92. 100 1.2
93. 100 1.2
94. 100 1.2
95. 100 1.2
96. 100 1.2
97. 100 1.2
98. 100 1.2
99. 100 1.2
100. 100 1.2



Bibliography.

In writing this thesis the following books have been used:-

M. Baire, "Sur les Fonctions de Variables Reelles."

Schoenflies, "Die Entwicklung der Lehre von den Punktmannigfaltigkeiten."

Dini, "Grundlagen für eine Theorie der Functionen einer Veränderlichen Reellen Grösse."

E. J. Townsend, "Ueber den Begriff und die Anwendung des Doppelimages."

E. J. Townsend, Lecture notes on Theory of Functions of Real Variable.

Introduction.

Suppose we have given the infinite series

$$u_1(x) + u_2(x) + u_3(x) + \dots + u_n(x) + \dots$$

each term of which is a continuous function of x , within the interval $\alpha < x < \beta$.

Let $S_n(x)$ represent the sum of a finite number (n) of these $u(x)$ functions. Since each $u(x)$ function is continuous the sum of a finite number, or $S_n(x)$, will be continuous. For if each $u(x)$ is continuous we have

$$\lim_{x \rightarrow x_0} u(x) = u(x_0)$$

for each x_0 between α and β . But since the limit of a sum is equal to the sum of the limits provided each limit exists we have

$$\lim_{x \rightarrow x_0} S_n(x) = \lim_{x \rightarrow x_0} u_1(x) + \lim_{x \rightarrow x_0} u_2(x) + \dots + \lim_{x \rightarrow x_0} u_n(x)$$

for any finite n and for $\alpha < x_0 < \beta$. Therefore $S_n(x)$ is a continuous function of x . Now if

$$\lim_{n \rightarrow \infty} S_n(x) = \phi(x),$$

then we say that the function $\phi(x)$ is equal to the infinite

series $\sum_{n=1}^{\infty} u_n(x)$, or $\phi(x)$ is representable by an infinite series each term of which is a continuous function of x ; that is, we may write

$$\phi(x) = u_1(x) + u_2(x) + u_3(x) + \dots + u_n(x) + \dots$$

for $\alpha < x < \beta$.

An example will be sufficient to show that the function $\phi(x)$ may be so represented by an infinite series and be discontinuous at the same time.

Consider the infinite series

$$\phi(x) = u_1(x) + u_2(x) + u_3(x) + \dots + u_n(x) + \dots$$

where $S_n(x) = (\sin \pi x)^{\frac{1}{2n+1}}$

$$\phi(x) = \sum_{n=0}^{\infty} S_n(x), \text{ for } \alpha \leq x \leq \beta.$$

Since $S_n(x)$ is a continuous function of x for every finite value of n , each $u(x)$ in the infinite series is also continuous in x . Since the sine of any angle which is a multiple of π is zero, we have

$$\text{for } x = 0, 1, 2, 3, \dots$$

$$\sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}} = \sum_{n=0}^{\infty} (0)^{\frac{1}{2n+1}} = 0.$$

But for $2m < x < 2m+1$, and $m = 0, 1, 2, \dots$, we have

$$\sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}} = (\sin \pi x)^0 = +1,$$

since $\sin \pi x$ is here always positive and greater than zero. And for $2m+1 < x < 2m+2$, $m = 0, 1, 2, \dots$, we have

$$\lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} (\sin \pi x)^{\frac{1}{2n+1}} = (\sin \pi x)^0 = -1,$$

since $\sin \pi x$ is here always negative. Then $\phi(x)$ is discontinuous at the points $x = 0, 1, 2, 3, 4, \dots$ and is still representable by an infinite series of continuous functions.

The question at once arises how many discontinuities may the function $\phi(x)$ have within a finite interval and still be so represented by an infinite series of continuous functions. Prof. E. J. Townsend has made a study of this problem.

[See his thesis, "Ueber den Begriff und die Anwendung des Doppellimes."] He started out by defining the $u(x)$ as continuous in x and upon this hypothesis has shown that $\phi(x)$ can at most be only pointwise discontinuous. We propose the converse problem, namely can every pointwise discontinuous function be so represented. That is, we shall study

the sufficient conditions which must be placed upon the function $\phi(x)$ in order that it can be represented by an infinite series each term of which is a continuous function of x ; or in other words, what conditions must a given function $\phi(x)$ satisfy in order that we may write

$$\phi(x) = u_1(x) + u_2(x) + u_3(x) + \dots + u_n(x) + \dots,$$
where $\alpha < x < \beta$ and where each $u(x)$ is a continuous function.

M. René Baire has discussed this problem in his thesis, "Sur les Fonctions de Variables Réelles", and has there announced the following conclusion: "In order to be represented by an infinite series of continuous functions it is sufficient that the given function $\phi(x)$ be at most pointwise discontinuous with respect to every perfect assemblage."

However, M. Baire has studied the problem from the standpoint of the theory of assemblages

while I propose to make the properties of limits the basis of my investigation of this problem. I shall also confine my discussion to functions of a real variable.

In our investigation, it will often be convenient to have a geometric representation of our problem. [See Figure I.] Since $S_n(x)$ is a function of both x and n , if we put $y = \frac{1}{n}$ then $S_n(x)$ becomes $f(x, y)$ and we may use the Cartesian system of coordinates to give a geometric representation of our problem. Since n is defined for only the positive integral numbers, y is defined for only the values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$

which are dense at $y = 0$.

Before we can write

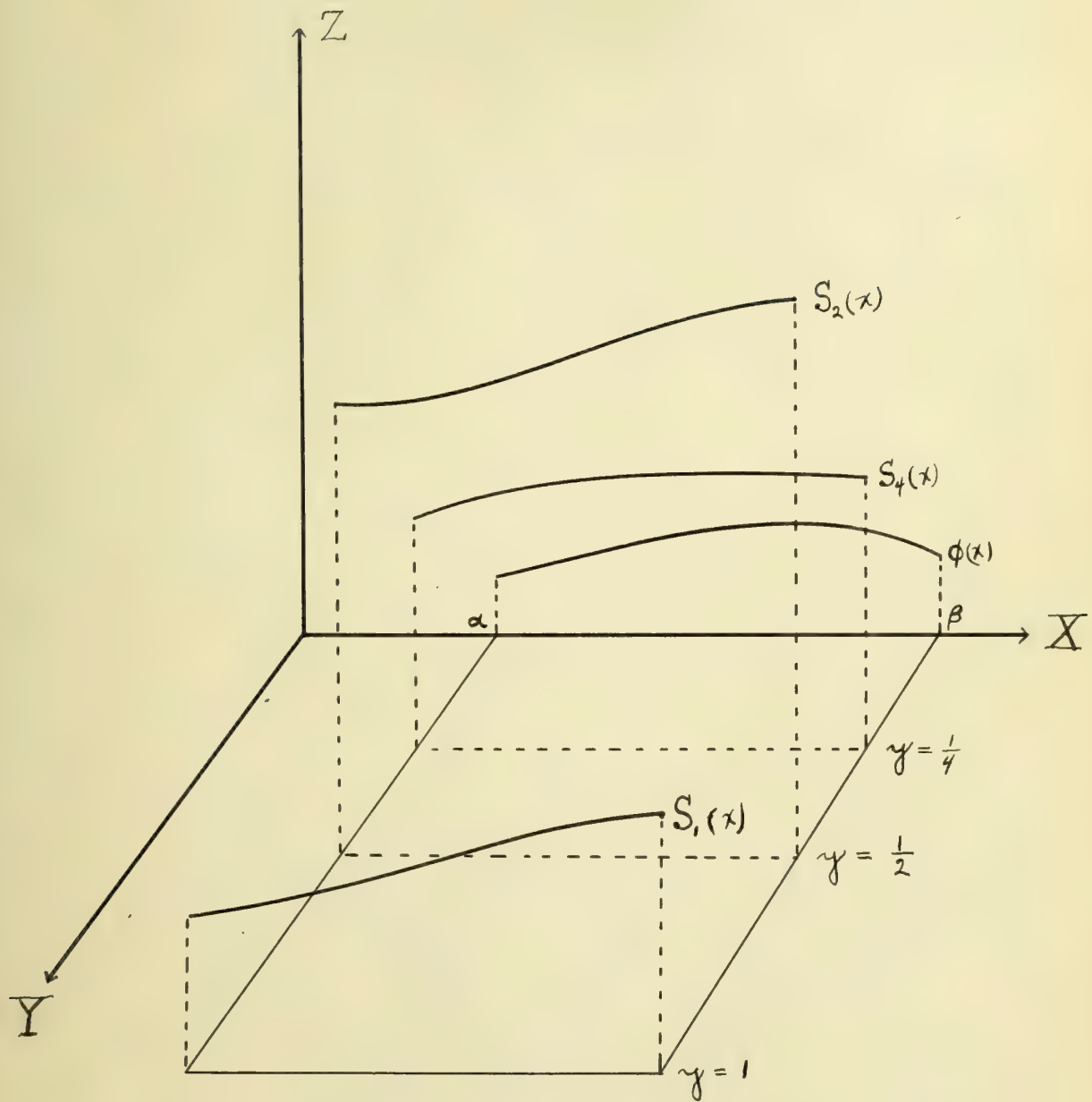
$$\phi(x) = u_1(x) + u_2(x) + u_3(x) + \dots + u_n(x) + \dots$$

two conditions must be satisfied;

first

$$\lim_{x \rightarrow x_0} S_n(x) = \lim_{x \rightarrow x_0} f(x, y_n) = f(x_0, y_n) \quad (1)$$

Figure I



for $\alpha < x_0 < \beta$ and for $0 < y_0 \leq 1$, which makes each $u(x)$ function continuous; second

$$\lim_{n \rightarrow \infty} S_n(x_0) = \lim_{y \rightarrow 0} f(x_0, y) = \phi(x_0) \quad (2)$$

for each $\alpha < x_0 < \beta$, which says that the infinite series, for each x , converges toward $\phi(x)$. One problem

then is how may $\phi(x)$ be defined in order that it be possible to find an $f(x, y)$, defined for area $\alpha \leq x \leq \beta$, $0 < y \leq 1$, which satisfies conditions ① and ② above.

We obtain our infinite series from the $f(x, y)$ by giving y successively the values

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$$

This is a special case of the following more general problem: How may $\phi(x)$ be defined in order that it be possible to find, in area $\alpha \leq x \leq \beta$, $0 < y < 1$, a function $f(x, y)$ which is continuous in x alone and continuous in y alone, such that $\phi(x) = f(x, y)$ for each x within interval of definition.

10
We shall investigate this
more general problem stopping
at times to make appli-
cations to infinite series.

§1. We shall first consider the case where $\phi(x)$ is a continuous and single valued function of x and show that it is always possible to represent such a function by an infinite series of continuous functions.

Since $\phi(x)$ is continuous we have

$$\lim_{x \rightarrow x_0} \phi(x) = \phi(x_0), \quad \alpha \leq x_0 \leq \beta.$$

Let $f(x, y)$ be defined by the relation

$$f(x, y) = (1-y)\phi(x) + y\psi(x), \quad (2)$$

where $\phi(x)$ is the given continuous function and where $\psi(x)$ is any arbitrary, continuous singlevalued function; that is, we define $f(x, y)$ linearly between $\phi(x)$ and $\psi(x)$. We have now to show that $f(x, y)$ as thus defined satisfies conditions (1) and (2) of page 7. Since $\phi(x)$ and $\psi(x)$ are each continuous by hypothesis, the sum of each multiplied by a constant will be continuous and we have

$$\lim_{x \rightarrow x_0} f(x, y_2) = \lim_{x \rightarrow x_0} [(1-y_2)\phi(x) + y_2\psi(x)] = f(x_0, y_2),$$

for each $\alpha \leq x_0 \leq \beta$, $y_2 \geq 0$. Hence

condition ① is satisfied. We also have

$\lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} [(1-y)\phi(x_0) + y\psi(x_0)] = \phi(x_0)$
and condition ② is satisfied. Therefore when $\phi(x)$ is continuous, it is representable by an infinite series of continuous functions.

It may be of interest to note here that Prof. Townsend has shown in his thesis mentioned above [see his thesis page 64] that, "If the terms of an infinite series are continuous functions of x and the series converges uniformly toward $\phi(x)$, then $\phi(x)$ is a continuous function."

He, however, calls attention to the fact that this is only a sufficient condition for the continuity of $\phi(x)$, for $\phi(x)$ may be continuous without the infinite series which it represents being uniformly convergent.

Example:- Given the continuous function $\phi(x) \equiv x+2$, to find an infinite series of continuous functions which it represents.

Arbitrarily take $\psi(x) = 3x + 2$. Then relation (3) of page 11 gives us

$$f(x, y) \equiv (1-y)(x+2) + y(3x+2) = x + 2xy + 2.$$

Hence we have

$$S_1(x) = 3x + 2$$

$$w_1(x) = 3x + 2$$

$$S_2(x) = 2x + 2$$

$$w_2(x) = -x$$

$$S_3(x) = \frac{5}{3}x + 2$$

$$w_3(x) = -\frac{x}{3}$$

$$S_4(x) = \frac{3}{2}x + 2$$

$$w_4(x) = -\frac{x}{6}$$

$$S_{n-1}(x) = \frac{n+1}{n-1}x + 2$$

$$S_n(x) = \frac{n+2}{n}x + 2$$

$$w_n(x) = -\frac{2}{n(n-1)}x$$

Therefore

$$\phi(x) \equiv (x+2) = (3x+2) - x - \frac{x}{3} - \frac{x}{6} - \frac{x}{10} - \frac{x}{15} - \frac{x}{n(n-1)}$$

Since $\psi(x)$ is any arbitrary continuous function and I may be selected in an infinite number of ways we see that any given continuous function $\phi(x)$ may be represented by an infinite series of continuous functions in an infinite number of ways.

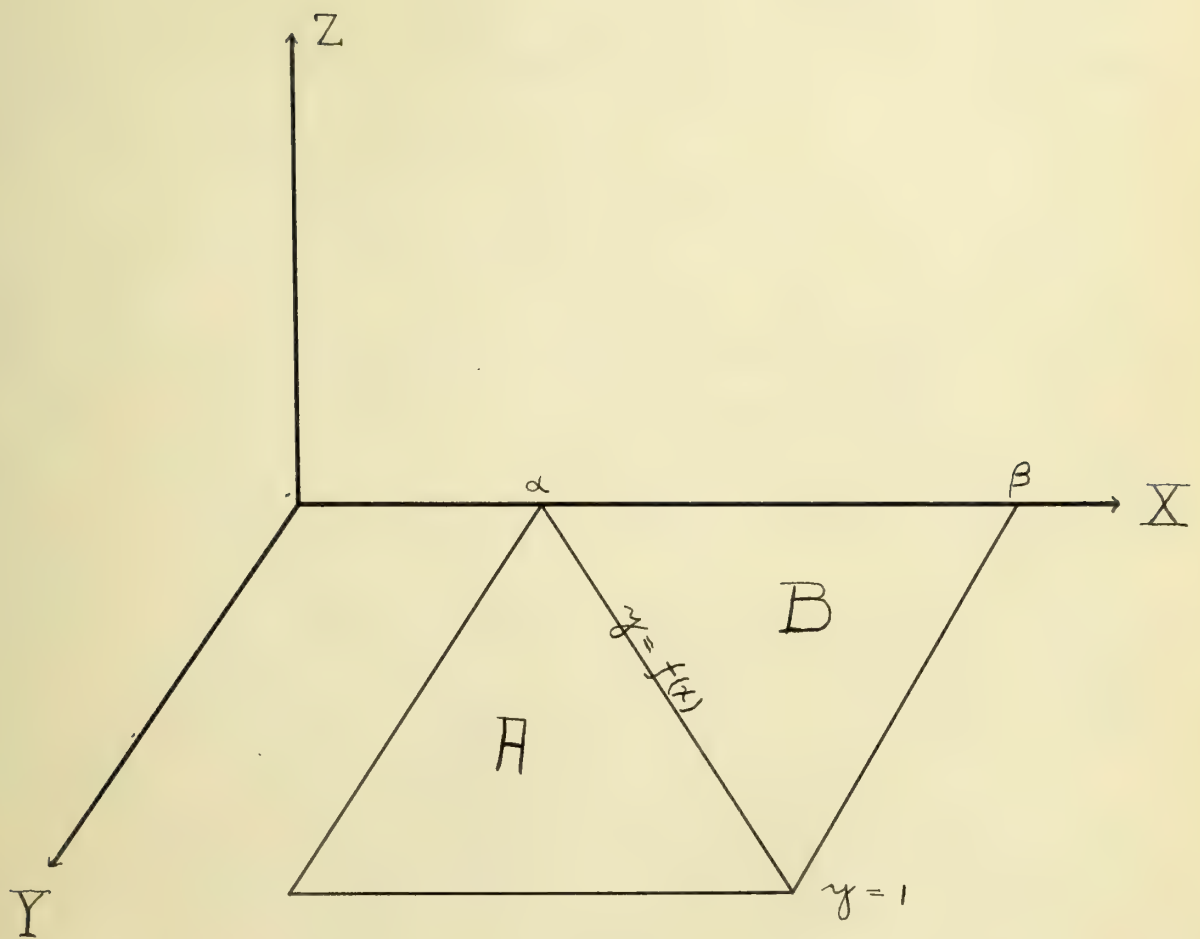
§2. We shall next consider the case where $\phi(x)$ has a single point of discontinuity, say at $x = \alpha$, and show that such a $\phi(x)$ is representable by an infinite series of continuous functions. In each case we shall consider only the area $\alpha \leq x \leq \beta$, $0 \leq y \leq 1$. By hypothesis then we have

$$\lim_{x \rightarrow \alpha} \phi(x) \neq \phi(\alpha)$$

$$\lim_{x \rightarrow x_0} \phi(x) = \phi(x_0) \quad , \quad \text{for } \alpha < x_0 \leq \beta.$$

Draw the lines $x = \alpha$, $x = \beta$, and $y = 1$ [see Figure II.]. Through the point $(\alpha, 0)$ draw any continuous curve, $y = f(x)$, say the diagonal of the rectangle. This curve divides the area of definition into two areas which we shall call area A and area B. Within area A we define $f_A(x, y)$ as any function continuous in x alone and in y alone, boundaries included, such that $f_A(\alpha, 0) = \phi(\alpha)$. Within area B we define $f_B(x, y)$ so that it shall vary linearly between

Figure II



$f_A(x, y)$ on the curve $y = f(x)$ and the limiting function $\phi(x)$. The function $f(x, y)$ is now completely defined for the whole rectangle, the boundary included and it remains to show that $f(x, y)$ as defined satisfies conditions ① and ② of page 7.

Since $f(x, y)$ was defined linearly in B and as continuous in x alone in A , we have

$$\lim_{x \rightarrow x_0} f(x, y_0) = f(x_0, y_0),$$
 for $\alpha \leq x_0 \leq \beta$, and $0 < y_0 \leq +1$. That is at every point of rectangle where $y > 0$ we have continuity in x and sufficient condition ① is satisfied.

Since $f(x, y)$ was defined as continuous in y alone in A and linearly in B , we have

$$\lim_{y \rightarrow 0} f(x_0, y) = f(x_0, 0) = \phi(x_0),$$
 for $\alpha < x_0 \leq \beta$. Since $f_A(x, 0)$ was defined to be equal to $\phi(x)$, we have

$$\lim_{y \rightarrow 0} f(x, y) = f_A(x, 0) = \phi(x).$$
 That is, we have continuity in y for every constant x . Therefore, since the two sufficient conditions are

17
satisfied, the given $\phi(x)$ may be represented by an infinite series of continuous functions.

Example:- Given the discontinuous function $\phi(x)$, where

$$\phi(x) = 0, \quad \text{for } x = 1$$

$$\phi(x) = \frac{1}{1-x}, \quad \text{for } 0 \leq x < 1,$$

to find an infinite series of continuous functions which will represent it.

$\phi(x)$ is continuous except at the point $x = 1$. There we have

$$\lim_{x \rightarrow 1} \phi(x) = \lim_{x \rightarrow 1} \frac{1}{1-x} = \frac{1}{0} = \infty \neq \phi(1) = 0.$$

Yet we have for $0 \leq x < 1$

$$\phi(x) = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

Here $\phi(x)$ has a single point of discontinuity at $x = 1$ and still is representable by an infinite series of continuous functions.

Remark. It should be noticed that, if $f(x, y)$ leads to a series which represents a given $\phi(x)$, then we are also led to a series which represents $\phi(x)$ by

$$F(x, y) \equiv f(x, y) + f_1(x, y)$$

provided $f_1(x, y)$ is continuous in x

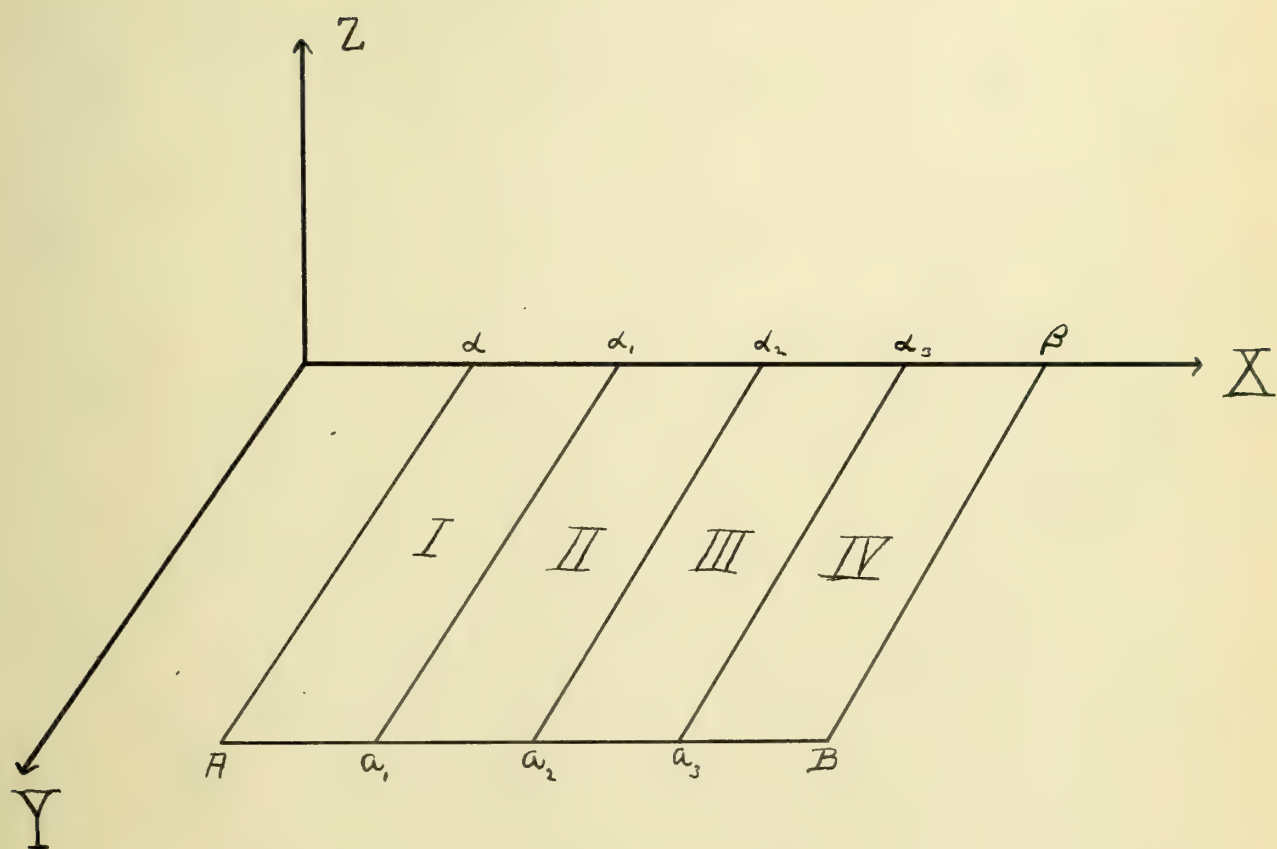
10
along and in y alone at every point within area of definition and provided $f_1(x, y) = 0$ for $y = 0$ and every $a \leq x \leq b$. $F(x, y)$ is continuous in x alone and in y alone and therefore satisfies the two sufficient conditions of page 7 and so is a solution. We may so select the $f_1(x, y)$ as to make $F(x, y)$ take any arbitrary set of continuous values upon three sides of our rectangle which includes $f(a, 0)$ and $f(b, 0)$; that is, on all sides except the side $y = 0$. We shall make use of this device later in our discussion.

§3. We shall next show that $\phi(x)$ is representable by an infinite series of continuous functions when $\phi(x)$ has a finite number of discontinuities within the interval (α, β) .

Let the points of discontinuity of $\phi(x)$ be $\alpha, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$, where n is finite. [See Figure III.] The lines $x = \alpha_1, x = \alpha_2, \dots, x = \alpha_n$ divide the rectangle into a finite number of areas. For each of the intervals $(\alpha, \alpha_1), (\alpha_1, \alpha_2), \dots, (\alpha_n, \beta)$, $\phi(x)$ is representable by the last section, since within each of these intervals there is but a single point of discontinuity. We have to show that there is a solution for the whole interval (α, β) .

Let us consider areas I and II. Let $f_1(x, y)$ be a solution in I and let $f_2(x, y)$ be a solution in II. On the line $x = \alpha_1$, select any set of continuous values including $\phi(\alpha_1)$. Then we can find some $f''(x, y)$ continuous in x alone and in y alone, such that, because of remarks on page

Figure III



17, we have

$$F_1(x, y) = f_1(x, y) + f_{11}(x, y)$$

as a solution in area I having the above arbitrary set of continuous values on the line $x = x_1$. Likewise we can find a $f_{22}(x, y)$ which is continuous in x alone and in y alone, such that

$$F_2(x, y) = f_2(x, y) + f_{22}(x, y)$$

will be a solution in II but having the ~~same~~ arbitrary set of continuous values on the line $x = x_1$. On the line $x = x_1$, then, $F_1(x, y)$ and $F_2(x, y)$ are equal.

Since $F_1(x, y)$ is a solution for area I; since $F_2(x, y)$ is a solution for area II; and since $F_1(x, y) = F_2(x, y)$ on the line $x = x_1$, we have continuity in y alone for the whole area (I + II). Moreover, $F_1(x, y)$ is continuous in x alone in area I because a solution in that area, $F_2(x, y)$ is likewise continuous in x alone in area II, and on the line $x = x_1$, $F_1(x, y)$ and $F_2(x, y)$ are equal; therefore we have continuity in x alone in every

point of area (I+II) where $y \neq 0$. We have then our sufficient conditions, namely, continuity in x and y separately, satisfied for the area (I+II), and $\phi(x)$, with two points of discontinuity, is representable for the interval (x, x_2) by an infinite series of continuous functions.

By considering in a similar way areas (I+II) and III we find $\phi(x)$ to be representable for interval (x, x_3) when having three points of discontinuity. By repeating this operation a finite number of times we finally have $\phi(x)$, with a finite number of discontinuities, representable for the entire interval $(x, 3)$.

We may restate the results of this section as follows:- The function $\phi(x)$ being defined as a single valued function for the interval $x \leq x \leq 3$, if the interval $(x, 3)$ be divided into a finite number of sub-intervals by the points $x_1, x_2, x_3, \dots, x_n$ such

that within each sub-interval,
 $x_i \leq x \leq x_{i+1}$, the function $\phi(x)$ is
 representable, then $\phi(x)$ is represent-
 able for the entire interval (a, b) .

Example:- The function $\phi(x)$, where

$$\phi(x) = \sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}},$$

 for $0 \leq x \leq v$, v being finite, is an example
 of a function having a finite
 number v discontinuities but still
 representable by an infinite series
 of continuous functions.

For $x = 0, 1, 2, \dots, v$, we have

$$\sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}} = \sum_{n=0}^{\infty} (0)^{\frac{1}{2n+1}} = 0.$$

For $2m < x < 2m+1$, and $m = 0, 1, 2, 3, \dots$
 we have

$$\sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}} = (\sin \pi x)^0 = +1.$$

For $2m+1 < x < 2m+2$, and $m = 0, 1, 2, \dots$
 we have

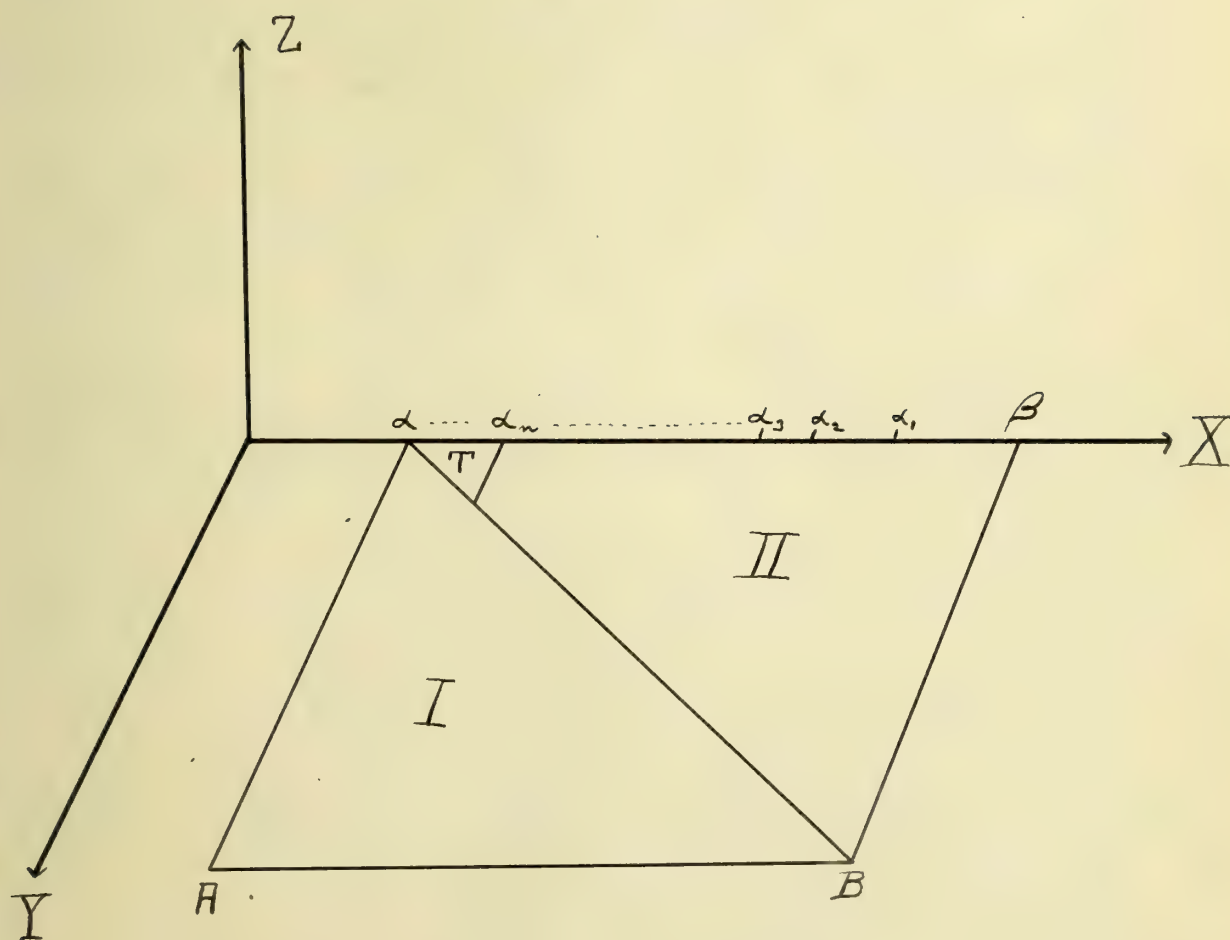
$$\sum_{n=0}^{\infty} S_n(x) = \sum_{n=0}^{\infty} (\sin \pi x)^{\frac{1}{2n+1}} = (\sin \pi x)^0 = -1.$$

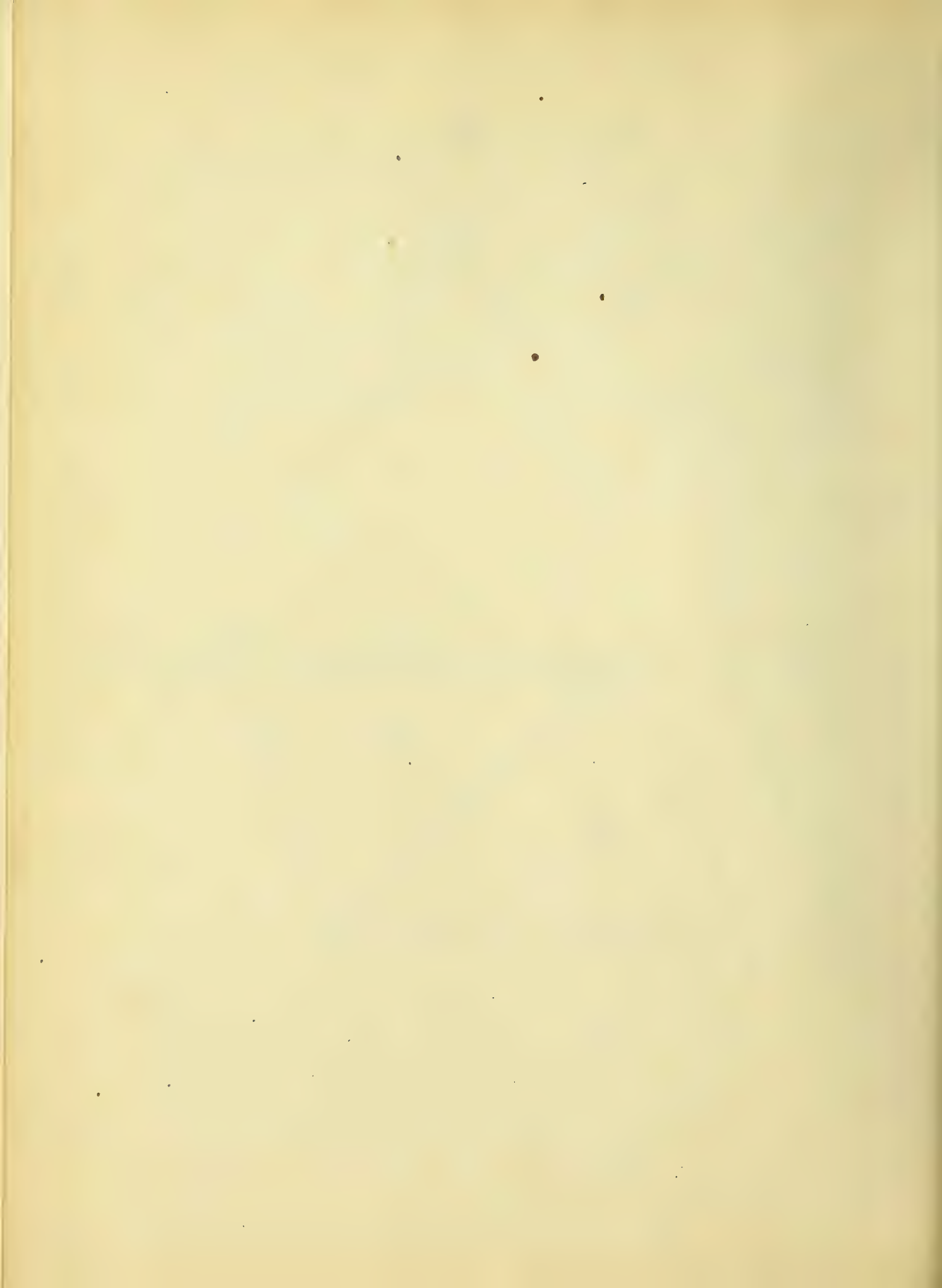
Therefore $\phi(x)$ is discontinuous at
 the finite set of points $0, 1, 2, 3,$
 $4, 5, \dots, v$ and is still representable
 by an infinite series.

27
§4. Now let $\phi(x)$ have an infinite number of points of discontinuity but with a single limiting point. We shall next show that $\phi(x)$, as thus defined, is representable by an infinite series of continuous functions.

Let $\phi(x)$ be discontinuous at the points $\alpha, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n, \dots$ which are dense at α . [See Figure IV.] Draw the lines $x = \alpha$, $x = \beta$, $y = 1$ which form our rectangle. Draw the diagonal and the line $x = \alpha_n$ until it intersects the diagonal. This divides the rectangle into three parts, which we will call I, II and T. In area I let $f_1(x, y)$ be any function continuous in x alone and in y alone having $f_1(\alpha, 0) = \phi(\alpha)$. Between α_n and β , $\phi(x)$ has only a finite number of discontinuities and therefore, by §3, $\phi(x)$ is representable in area II. Let $f_2(x, y)$ be such a solution for area II. Making use of remarks on page 17, let us

Figure IV





find some function $f_2(x, y)$, continuous in x alone and in y alone with $f_2(x, 0) = 0$, such that

$$F_1(x, y) \equiv f_1(x, y) + f_2(x, y)$$

will be a solution for area I and such that on line $\alpha\beta$ we have

$$F_1(x, y) = f_2(x, y).$$

We now have $f(x, y)$ defined for areas I and II. By taking n sufficiently large we can make the area T smaller than any arbitrary quantity. Therefore $f(x, y)$ is defined for the whole rectangle and it remains to show that $f(x, y)$, as thus defined, is continuous in x alone and in y alone.

In area I, $f(x, y) = F_1(x, y)$ which was defined as continuous in x alone. In area II, $f(x, y)$ was representable by $f_2(x, y)$ because in (α, β) there were only a finite number of discontinuities and, since $f_2(x, y)$ is a solution which satisfies the sufficient conditions, it must be continuous in x alone.



Since $F_1(x, y) = f_2(x, y)$ on the line αB , we have continuity in x on this diagonal. If we take any point in area I , we can bring such a point into area II by taking n sufficiently large; but in II there is continuity in x alone at every point where $y > 0$. Therefore, in the whole rectangle we have $f(x, y)$, as defined, continuous in x alone.

Since $F_1(x, y)$ is continuous in y alone, with $F_1(\alpha, 0) = f_1(\alpha, 0) = \phi(\alpha)$, we have

$$\lim_{y \rightarrow 0} f(\alpha, y) = \lim_{y \rightarrow 0} F(\alpha, y) = \phi(\alpha),$$

that is, we have continuity in y for $x = \alpha$. Since $\phi(x)$ is representable by $f_2(x, y)$ for interval (α, β) , there must be continuity in y alone in area II and we have

$$\lim_{y \rightarrow 0} f(x_0, y) = \lim_{y \rightarrow 0} f_2(x_0, y) = \phi(x_0),$$

for each $\alpha < x_0 \leq \beta$. We have then continuity in y alone for each constant x_0 within the interval (α, β) , and points included. The two sufficient con-

ditions, namely continuity in x alone and in y alone, are thus satisfied for the entire area of definition and therefore $\phi(x)$, when discontinuous at an infinite set of points with a single limiting point, may be represented by an infinite series of continuous functions.

Example:- The function

$$\phi(x) = \sum_{n=1}^{\infty} S_n(x) = \sum_{n=1}^{\infty} \frac{(1+x \sin \frac{\pi}{x})^n - 1}{(1+x \sin \frac{\pi}{x})^n + 1}$$

for $0 \leq x \leq 1$, is an example of a function which is discontinuous at an infinite set of points M having a single limiting point.

At each point of M , where $M = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots$,

$$\sum_{n=1}^{\infty} S_n(x) = \sum_{n=1}^{\infty} \frac{(1+x \sin \frac{\pi}{x})^n - 1}{(1+x \sin \frac{\pi}{x})^n + 1} = \sum_{n=1}^{\infty} \frac{1^n - 1}{1^n + 1} = 0.$$

For $\frac{1}{2n} < x < \frac{1}{2n+1}$, we have

$$\sum_{n=1}^{\infty} S_n(x) = \sum_{n=1}^{\infty} \frac{1 - \frac{1}{(1+x \sin \frac{\pi}{x})^n}}{1 + \frac{1}{(1+x \sin \frac{\pi}{x})^n}} = \frac{1-0}{1+0} = +1.$$

For $\frac{1}{2n+1} < x < \frac{1}{2n+2}$, we have

$$\sum_{n=1}^{\infty} S_n(x) = \sum_{n=1}^{\infty} \frac{(1+x \sin \frac{\pi}{x})^n - 1}{(1+x \sin \frac{\pi}{x})^n + 1} = \frac{0-1}{0+1} = -1.$$

Therefore $\phi(x)$ is discontinuous at the points

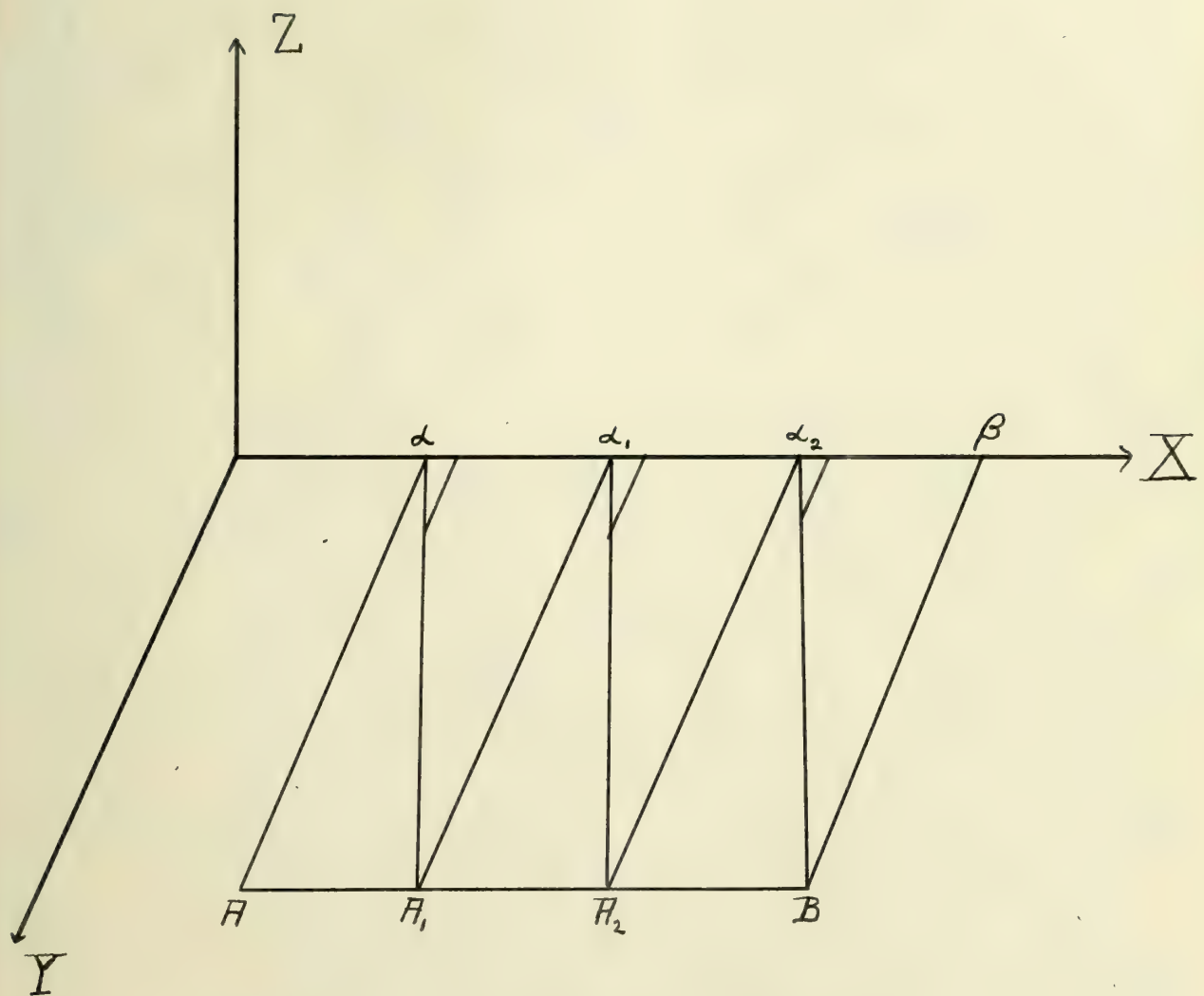
$$M \equiv 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots$$

which are dense at $x = 0$. We notice that $S_n(x)$ is continuous for any constant value of n . Hence $\phi(x)$, as here defined, may be represented by an infinite series of continuous functions.

§5. Let $\phi(x)$ be discontinuous at an infinite assemblage of points M and let the first derived assemblage M' be finite. We shall next show that $\phi(x)$ as thus defined may be represented by an infinite series of continuous functions.

Let the M' points be $d, d_1, d_2, d_3, \dots, d_n$, there being a finite number of them. [See Figure V.] These points divide the interval (α, β) into a finite number of sub-intervals. Within each one of these sub-intervals there is an infinite number of M points, but having just one limiting point. Therefore, by the last section, $\phi(x)$ is representable within each of these sub-intervals. But there is a finite number of these sub-intervals within which $\phi(x)$ is representable. Hence, by result of section three, $\phi(x)$ is representable within the total interval (α, β) .

Figure V



32

A slight modification of the above leads to a more general case and enables us to extend our class of representable functions. In the interval (α, β) , let there be a finite number of points $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$. Surround each of these points by an arbitrarily small interval ϵ_n . Let ϵ be the upper limit of $\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n$. Since there is a finite number of these small intervals, we have

$$\lim_{\epsilon \rightarrow 0} (\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots + \epsilon_n) = 0.$$

For each sub-interval (α_i, α_{i+1}) , we may find a solution as we did in section four. Since there is a finite number of these sub-intervals within which $q(x)$ has a solution, then by section three, there is a solution for the whole interval (α, β) . Therefore we may state the following general case: If within the interval (α, β) we have a discrete^{*} assemblage of

* See definition page 33.

points $\alpha, \alpha_1, \alpha_2, \dots$ enclosed in a finite set of arbitrarily small intervals ϵ , and if $\phi(x)$ is representable in each remaining interval then $\phi(x)$ is representable for the whole interval (α, β) .

Let $\phi(x)$ be discontinuous at a set of points M which is discrete. Then by above theorem such a $\phi(x)$ is representable by an infinite series of continuous functions.

Note. By a discrete assemblage we mean an assemblage which may be enclosed within a finite number of arbitrarily small intervals the limit of whose sum is zero.

34
§6. The results of the last section enable us to show that $\phi(x)$ is representable by an infinite series of continuous functions when the assemblage M of discontinuities has an infinite number of limiting ^{points}. First consider the case where M' is infinite but M'' is finite. Here the set $(M' + M'')$ is discrete and may be enclosed in a finite set of arbitrarily small intervals. Since in the remaining intervals of (α, β) there is only a finite number of points of M , $\phi(x)$ will be representable in each remaining interval. Therefore, by the result of section five, $\phi(x)$ is, in this case, representable for the whole interval (α, β) .

Consider the case where M , M' , and M'' are each infinite, but M''' is finite. Since the assemblage $(M' + M'' + M''')$ is discrete, we again see by theorem of last section that $\phi(x)$ is representable for the whole interval (α, β) in this

case.

In general, take the case where $M', M'', M''', \dots, M^{v-1}$ are each infinite but where M^v is finite, v also being finite. Here the assemblage

$$M' + M'' + M''' + \dots + M^v$$

is discrete, and when its points are cut out by a finite set of arbitrarily small intervals,

$\phi(x)$ is representable in each remaining interval, because these intervals contain only a finite number of M points. Therefore, again we have, by the theorem of the last section, $\phi(x)$ representable for the entire interval (α, β) .

Since v was taken finite we have only a finite number of derived assemblages. Therefore, M is an assemblage of the first species. We have then up to this point shown that a given $\phi(x)$ is representable by an infinite series each term

of which is a continuous fraction
of x , when $\phi(x)$ is continuous or
when the points of discontinuity
of $\phi(x)$ form an assemblage
of M of the first species.

§7. Let us now consider the case where the points of discontinuity of $\phi(x)$ constitute a set of the second species. Let M be the assemblage of discontinuities of $\phi(x)$ as before. Let M have an infinite number of derived assemblages but let M^w be finite, w being Cantor's first transfinite number. Now we have only to show that the assemblage

$Q \equiv M' + M'' + M''' + \dots + M^w + \dots + M^w$ is discrete and apply theorem of section five.

Consider the assemblage Q . The last term M^w is finite by hypothesis and can be enclosed within a finite set of arbitrarily small intervals. For the points not enclosed we have $M^w = 0$ and consequently there must be some finite number v for which $M^{v+1} = 0$ for the points still not enclosed. Hence we have left, after enclosing M^w , the assemblage $M' + M'' + M''' + \dots + M^v$.

But M^v is finite because it has no derived set, $M^{v+1} = \emptyset$. Therefore we can enclose M^v in a finite set of arbitrarily small intervals and we have still unenclosed the points

$$M' + M'' + M''' + \dots + M^{v-1},$$

where M^{v-1} is finite. We can then enclose M^{v-1} in a finite set of arbitrarily small intervals and we have left still unenclosed

$$M' + M'' + M''' + \dots + M^{v-2},$$

where M^{v-2} is finite. By continuing this operation we finally get all of the points of Q enclosed in a finite number of finite assemblages which is a finite assemblage. Therefore Q is discrete. By applying now the theorem of section five, we have $\phi(x)$ representable when M^w of the M discontinuities of $\phi(x)$ exists but is finite. When $M^w = 0$ the case becomes identical with that of section six.

If M^w is infinite but



M^{w+1} is finite we have

$$Q \equiv M' + M'' + M''' + \dots + M^{(w)} + \dots + M^w + M^{w+1}$$

which may be shown to be discrete by same method used above except that it requires one more step. This process may be continued indefinitely.

In general let α be any second class number,* and let $M^\alpha = 0$. Then we have

$$Q \equiv M' + M'' + \dots + M^{(n)} + \dots + M^w + \dots + M^{2w} + \dots + M^{\alpha-1},$$

where $M^{\alpha-1}$ is finite. Here again Q may be enclosed in a finite number of finite assemblies of small intervals and is therefore discrete. Then by theorem of section five, $\phi(x)$ is representable when $M^\alpha = 0$, α being a number of either first or second class.

Schoenflies has shown in his book on the theory of assemblies (See page 68 of his book) that if M is an enumerable and a closed set, then $M^\infty = 0$, and there exists a first α of the first or second class such that $M^\alpha = 0$. We

* See definition of second class numbers on page 40.

shall make use of this fact. Let M be enumerable and closed (If M is not closed consider set $M + M'$ which is closed). Then $M^\omega = 0$ and there exists an α , first or second class, for which $M^\alpha = 0$. Since $M^\alpha = 0$, Q is discrete and therefore $\phi(x)$ is representable by above proof. We have therefore shown that $\phi(x)$, when discontinuous at an enumerable and closed set of points M , is still representable by an infinite series of continuous functions.

Note. If we arrange all the ordinal numbers in order of magnitude we may divide them into two classes:-

1, 2, 3, 4, ----- ω ----- 1st class

ω , $\omega+1$, $\omega+2$, ----- 2nd class

A number of the first class then is an ordinal number which is formed by a single principle of formation, namely, the principle of addition.

A number of the second class needs two principles of formation, namely, the principle of addition and the prin-

iple of limits.

Example:- Let us consider the example where

$$\phi(x) = \lim_{n \rightarrow \infty} S_n(x)$$

$$S_n(x) = \sum_{K=1}^n \frac{1}{K!} (\cos K! \pi x)^{2^n}, \quad 0 \leq x \leq 1.$$

This series is a continuous function for each finite n and converges for each constant value of x within the given interval because for each constant x we have

$$\sum_{K=1}^{\infty} \frac{1}{K!} (\cos K! \pi x)^{2^n} \leq \sum_{K=1}^{\infty} \frac{1}{K!}$$

for all values of n and the series $\sum \frac{1}{K!}$ is known to be convergent.

Since this series, considered as a function in K and n , is uniformly convergent, we have

$$\lim_{n \rightarrow \infty} S_n(x) = \lim_{n \rightarrow \infty} (\cos \pi x)^{2^n} + \lim_{n \rightarrow \infty} \frac{1}{2!} (\cos 2! \pi x)^{2^n} + \dots + \lim_{n \rightarrow \infty} \frac{1}{K!} (\cos K! \pi x)^{2^n} + \dots$$

Now, if x is any rational number, that is, if $x = \frac{p}{q}$ where p and q are prime to each other, some term we will have $K!$ divisible by q . For every term preceding this one we have

$$(\cos K! \pi x)^{2^n} < 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{K!} (\cos K! \pi x)^{2^n} = 0.$$

But from this term on we have

$$(\cos K! \pi x)^{2^n} = 1.$$

Therefore, for every rational value of x , we have

$$\sum_{n=0}^{\infty} S_n(x) > 0.$$

On the other hand if x is irrational, however great we may take K , we always have

$$(\cos K! \pi x)^{2^n} < 1$$

$$\text{and } \sum_{n=0}^{\infty} \frac{(\cos K! \pi x)^{2^n}}{K!} = 0.$$

Hence this series represents a function which is greater than zero at all rational points and equal to zero at all irrational points. It, therefore, gives a function which is defined by a series of continuous functions but is discontinuous at all rational points, that is, at an enumerable set of points.

§8. We now wish to consider the cases where $\phi(x)$ is discontinuous at an assemblage M of points which is non-enumerable. In section five we have shown that $\phi(x)$ was always representable by a finite series of continuous ^{functions} when M was discrete. Now some non-enumerable assemblages are discrete, that is, have the content zero. Therefore, by the result of section five, if $\phi(x)$ is discontinuous at a set of points which is non-enumerable but discrete, then $\phi(x)$ is representable by a series of continuous functions.

47
§ 9. Consider now the non-enumerable assemblages which are perfect and in no sub-interval everywhere dense. Let $\phi(x)$ be discontinuous at a set of points M which is perfect and in no sub-interval everywhere dense. Before considering this case we shall notice an interesting property which we shall use later.

Remark. Let M be an assemblage which is perfect and in no sub-interval everywhere dense. Let $\phi(x)$ be defined only for the M points. Then we can discuss the continuity of $\phi(x)$ at each M point because each is a limiting point since M is perfect. If we think of $\phi(x)$ as being defined only at the M points and find it is continuous at each M point, then we say $\phi(x)$ is continuous in the domain M . We may now state the following theorem:-

If $\phi(x)$ is continuous with respect to the domain M , where M is perfect and in no sub-interval everywhere dense, then it is always

possible to find a $\psi(x)$ which is equal to the $\phi(x)$ at the M points and continuous for the entire interval (α, β) .

Let $\psi(x)$ be equal to $\phi(x)$ at the M points. Let A_n and B_n be any two contiguous points of M . Define $\psi(x)$ linearly between the values $\phi(A_n)$ and $\phi(B_n)$. We have $\psi(x)$ now defined for every point of the interval (α, β) and we have to show that $\psi(x)$ is continuous at every point. Let x_0 be any M point. Since $\phi(x)$ was defined as continuous in the domain M , to every arbitrarily small number ϵ there corresponds another number δ such that at every M point in the interval $(x_0 - \delta, x_0 + \delta)$ we have

$$|\phi(x) - \phi(x_0)| < \epsilon.$$

Let H be any point in interval $(x_0 - \delta, x_0 + \delta)$ which is not an M point. Point H will therefore be in some segment (A_n, B_n) which

is contiguous to M , that is, an interval ad between two adjoining M points. But $\psi(x)$ was defined linearly between points A_n and B_n , therefore $\psi(H)$ lies between $\psi(A_n)$ and $\psi(B_n)$, or between $\phi(A_n)$ and $\phi(B_n)$. We have then for every point of the interval the inequality

$$|\psi(x) - \psi(x_0)| < \epsilon,$$

which proves $\psi(x)$ to be continuous throughout the interval (α, β) . This property enables us to consider without any loss of generality a $\phi(x)$ which is zero at the M points; for suppose we put

$$\Phi(x) = \phi(x) - \psi(x),$$

where $\phi(x)$ is continuous in the domain M which is perfect and in no sub-interval everywhere dense, and where $\psi(x) = \phi(x)$ at the M points and is continuous throughout interval (α, β) . The function $\Phi(x)$ will possess all the properties as to continuity

47
of $\phi_1(x)$, since it differs from $\phi(x)$ by a continuous function, but $\phi(x)$ is zero at the M points. If $\phi(x)$ is representable of course $\phi_1(x)$ will also be representable.

By the use of this property we can extend our list of representable functions. Let M be the assemblage of discontinuities of $\phi(x)$ which are greater than some arbitrarily small positive number σ . Let M be perfect and in no sub-interval everywhere dense and let $\phi(x)$ be continuous in the domain M . We now propose to show that under these conditions, $\phi(x)$ is representable by an infinite series of continuous functions.

It should be noticed that $\phi(x)$ is discontinuous at the points M when considered with respect to the domain of the continuum (α, β) . In consequence of the remark made above, there will be no loss of generality in taking $\phi(x) = 0$ at the M points. Since M

is perfect and in no sub-interval everywhere dense we may think of M as being defined by the end points, plus their limiting points, of a set of intervals $B = \{B\}$ which is everywhere dense in the interval (α, β) (see Schauder's p. 76). Since the set B is fully ordered, that is has a first term, we may arrange them in the order of magnitude putting the largest one first, as

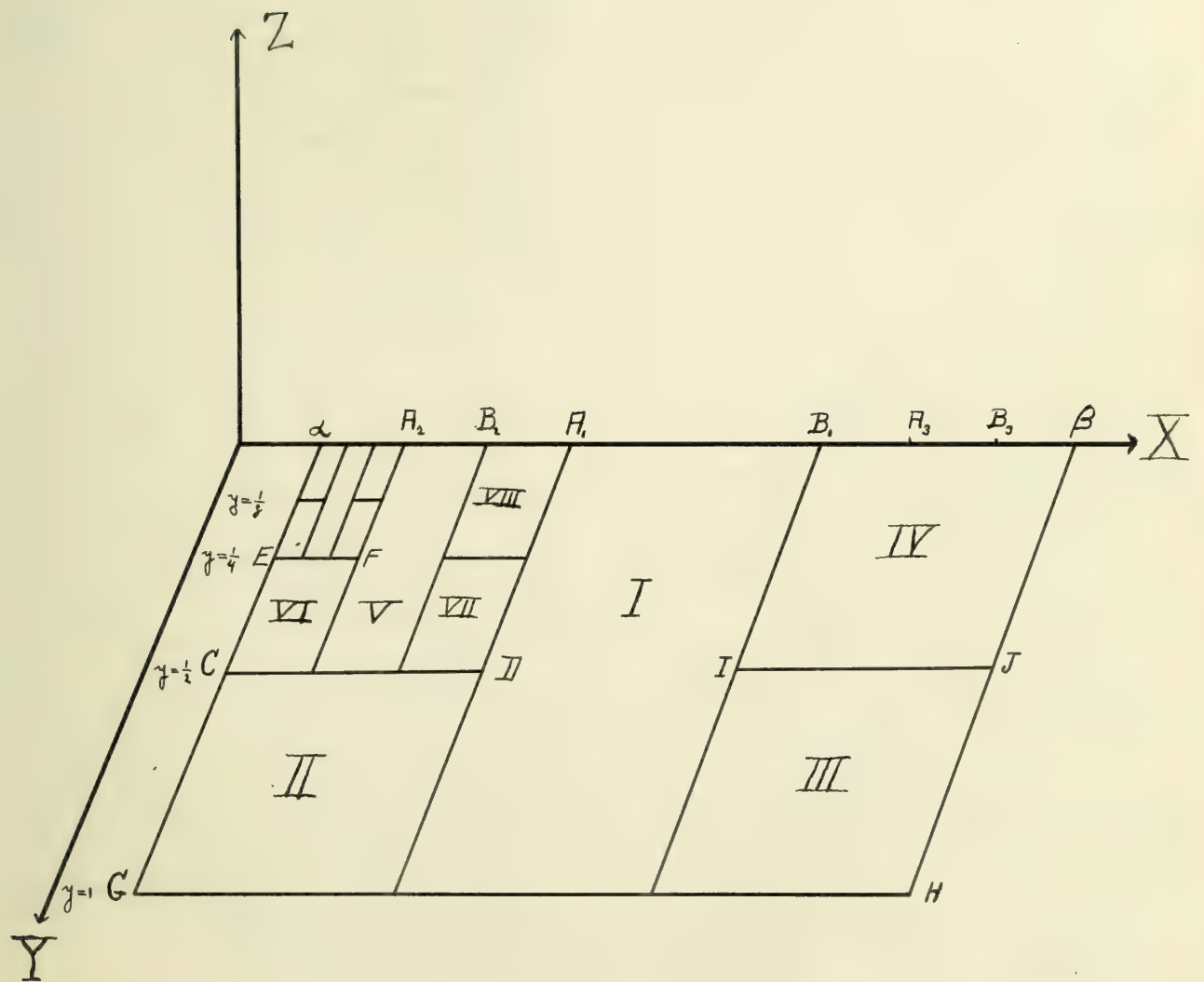
$$A_1 B_1, A_2 B_2, A_3 B_3, \dots, A_n B_n, \dots$$

Through the M points $[A_1, B_1, A_2, B_2, \dots]$ draw lines parallel to $x=0$ and on these lines define $f(x, y) = 0$. (See Figure VI.) Then on each of these lines we have continuity in y alone, ~~for~~ because

$$\lim_{y \rightarrow 0} f(x, y) = \lim_{y \rightarrow 0} 0 = 0$$

for each constant x_0 equal to $A_1, B_1, A_2, B_2, \dots$. The interval (A_1, B_1) contains no M points and therefore $\phi(x)$ is representable for such interval (A_1, B_1) ; let $f_1(x, y)$ be a solution for this interval. Draw

Figure VI



The lines $y = \frac{1}{2}$ [CD and IJ]. In areas II and III let $f_2(x, y) = 0$ and $f_3(x, y) = 0$ respectively. We then have $f(x, y)$ defined in areas I, II, and III in such a way that there is continuity in x alone and in y alone at every point.

Now treat area ($\alpha H, II C$) just as we treated area ($\alpha \beta H G$).

Since ($H_2 B_2$) contains no M points, $\phi(x)$ is representable in area V;

let $f_5(x, y)$ be such a solution.

Draw the line $y = \frac{1}{4}$ [EF] and

in areas VI and VII let $f(x, y) = 0$,

or $f_6(x, y) = 0$ and $f_7(x, y) = 0$. We

have then $f(x, y)$ defined in V, VI and VII in such a way that there is continuity in x alone and in y alone at every point.

Treat areas ($\alpha H_2 F E$), VIII and IX as we have treated areas ($\alpha \beta H G$) and ($\alpha H, II C$) and repeat this process indefinitely. After a finite number of such operations, we have for any

point where $y > 0$, our $f(x, y)$ defined in such a way that there is continuity in x alone. Remembering that on every line $y = x_0$, where x_0 is a point of M , $f(x, y)$ was defined to be zero, we see that at every point we have $f(x, y)$ continuous in y alone. We have then $f(x, y)$ defined as continuous in x alone and in y alone, at every point of area $(\alpha \beta \wedge \zeta)$ where $y > 0$. Therefore we may conclude that $\phi(x)$, when discontinuous at a set of points M which is perfect and in no sub-interval everywhere dense, and when $\phi(x)$ is continuous in the domain M , is representable by an infinite series of continuous functions.

§10. From the theory of assemblages we get the following two facts:

① If $M^{\alpha} = 0$, then M is enumerable, M^{α} being the derived assemblage of the α th order, α being the first transfinite number of the third class.

(See Schoenflies' "Punktmangfaltigkeiten" page 68.)

② If M is non-enumerable, then M^{α} is not zero but is a perfect assemblage, and M is so constituted that in each interval between the M^{α} points M is enumerable. (See Baire, "Sur les Fonctions de Variables Réelles", page 44 or "Le Mémoire de M. Borel" in volume II of Acta Mathematica.)

In section seven we saw that $\phi(x)$ was representable when discontinuous at an enumerable set of points. That result depended upon the fact that $\phi(x)$ was representable in each interval remaining after enclosing a certain set within a finite set of small intervals. Remembering this we see that the result of section seven may be stated as follows:-

③ If $\phi(x)$ is discontinuous at

an enumerable assemblage M and if $\phi(x)$ is representable in each interval between the M points, then $\phi(x)$ is representable for the entire interval (α, β) .

In obtaining the result of section nine we considered $\phi(x)$ continuous in each interval (A_n, B_n) between two adjacent points of M . The result would be unchanged if we considered $\phi(x)$ as representable in each interval (A_n, B_n) and we may therefore state the result of section nine in the following form:-

(4) If $\phi(x)$ is discontinuous at the set M which is perfect and in no sub-interval everywhere dense; if $\phi(x)$ is continuous in the domain M ; and if $\phi(x)$ is representable in each interval between the M points, then $\phi(x)$ is representable for the entire interval (α, β) .

By means of these four principles, we may now extend our results. In all that follows

57
 let M be the assemblage of points in interval (α, β) where $\phi(x)$ has a discontinuity greater than some number H . M is to be closed, that is, contains its limiting points. If M^{ω} is zero, then M is enumerable by principle (1) above and by result of section seven $\phi(x)$ is representable when discontinuous at an enumerable set of points. But if $M^{\omega} \neq 0$, then by principle (2) above M^{ω} is perfect and M is enumerable in every interval between the M^{ω} points. Since we are considering only discontinuities $> H$, then M^{ω} must be in no sub-interval everywhere dense. We may talk of continuity in domain M^{ω} because each point of M^{ω} is a limiting point. Then if $\phi(x)$ is continuous in the domain M^{ω} by principle (4) it is representable.

But suppose $\phi(x)$ is discontinuous in the domain M^{ω} . Let M_1 be the points of discontinuity of $\phi(x)$ with respect to the domain M^{ω} which discontinuities are greater

than some number A . If $M_1^{\omega} = 0$ then M_1 is enumerable. $\Phi(x)$ is representable in each interval between the M_1 points by principle (4) and therefore $\Phi(x)$ is, by principle (3), representable for the entire interval (α, β) . But if $M_1^{\omega} \neq 0$, it is perfect and in no sub-interval everywhere dense and, by principle (4), $\Phi(x)$ will then be representable for the whole interval (α, β) provided $\Phi(x)$ is continuous in the domain M_1^{ω} .

Again suppose $\Phi(x)$ is discontinuous in the domain M_1^{ω} . Let M_2 be the assemblage of these points of discontinuities with respect to the domain M_1^{ω} . Treat M_2 as we treated M_1 , and we see that if $M_2^{\omega} = 0$ then $\Phi(x)$ is representable for the whole interval (α, β) , because M_2 is enumerable and $\Phi(x)$ is representable in each interval contiguous to M_2 . If $M_2^{\omega} \neq 0$, it is perfect and in no sub-interval everywhere ^{dense}, because we are considering only the discontinuities $> A$. In this

case $\phi(x)$ is representable for the whole interval (α, β) provided $\phi(x)$ is continuous in the domain M_2^{α} .

Continuing this indefinitely we get a series of assemblages

$$M, M_1, M_2, M_3, \dots, M_v, \dots$$

each of which is a sub-assemblage of its preceding one and each of which is related to the preceding one as M_1 is related to M . If M_{v-1} is enumerable then

$$M_{v-1}^{\alpha} = 0, \text{ and therefore } M_v = 0; \text{ or if}$$

$$M_{v-1}^{\alpha} \neq 0 \text{ and } \phi(x) \text{ is continuous in}$$

$$\text{the domain } M_{v-1}^{\alpha}, \text{ again } M_v = 0. \text{ We}$$

may then say in general that $\phi(x)$ is representable in the whole interval (α, β) when $M_v = 0$, where v is any number of the first class.

This may be further extended into the transfinite region. Suppose there is no finite number v for which $M_v = 0$. Then we have

$$M > M_1 > M_2 > \dots > M_v > \dots$$

which approaches some limit M_w , where M_w is related to M_1 as

59

Cantor's derived assemblage of the w^{th} order M'' is related to the first derived assemblage M' . M_w is a sub-assemblage of M and is enumerable if $M_w^{\omega} = 0$ and $\phi(x)$ is representable by application of principle (3) above. If $M_w^{\omega} \neq 0$, but $\phi(x)$ is continuous in domain M_w^{ω} then $\phi(x)$ is representable by principle (4). In either case $M_{w+1} = 0$.

Continuing this process indefinitely we get the series of assemblages

$M, M_1, \dots, M_v, \dots, M_w, M_{w+1}, \dots, M_{2w}, \dots, M_{\alpha}, \dots$
 each of which is related to its preceding one and M_1 is related to M . If some $M_{\alpha-1}^{\omega} = 0$, for some α either first or second class, then $M_{\alpha-1}$ is enumerable and $\phi(x)$ is representable by principle (3). If $M_{\alpha-1}^{\omega} \neq 0$ but $\phi(x)$ is continuous in domain $M_{\alpha-1}^{\omega}$, then $\phi(x)$ is representable by principle (4). However if either $M_{\alpha-1}$ is enumerable or if $M_{\alpha-1}^{\omega}$ exists and $\phi(x)$ is continuous in domain $M_{\alpha-1}^{\omega}$, we always have

$M_\alpha = 0$. If $M_\Omega = 0$ there always exists some α , either first or second class number, for which $M_\alpha = 0$. We may conclude then that it is sufficient to have $M_\Omega = 0$ in order that $\phi(x)$ be representable by an infinite series each term of which is a continuous function of x .

§11. We have seen that when $M_2 = 0$ the function is representable by an infinite series of continuous functions. We now propose to show that when $\phi(x)$ is pointwise discontinuous with respect to every perfect assemblage within the interval (α, β) , then we always have $M_2 = 0$ and therefore $\phi(x)$, as defined, is representable.

Let $\phi(x)$ be pointwise discontinuous with respect to every perfect assemblage within the interval (α, β) . Let M be the assemblage of discontinuities of $\phi(x)$ which are greater than some number H . If M is enumerable then $M^2 = 0$ and M_1 , being zero surely $M_2 = 0$. Since M^2 is perfect then by hypothesis $\phi(x)$ is pointwise discontinuous with respect to M^2 . Let M_1 be the assemblage of discontinuities of $\phi(x)$ greater than some number H in the domain of M^2 . If M_1 is enumerable, then $M_1^2 = 0$ and

therefore $M_2 = 0$. But if M_1 is non-enumerable we have $M_1^{\omega} \neq 0$. Since M_1^{ω} is perfect $\phi(x)$ is pointwise discontinuous with respect to M_1^{ω} , noticing that we are still considering only the discontinuities which are greater than A . Let M_2 be the assemblage of discontinuities of $\phi(x)$ with respect to M_1^{ω} . If M_2 is enumerable then $M_2^{\omega} = 0$ and $M_2 = 0$. But if M_2 is non-enumerable, we have $M_2^{\omega} \neq 0$. Continue this process indefinitely; that is in general, let M_α be the assemblage of discontinuities of $\phi(x)$ with respect to $M_{\alpha-1}^{\omega}$, where α is a number of first or second class. Then if M_α is enumerable $M_\alpha^{\omega} = 0$, and therefore $M_\alpha = 0$. But if M_α is non-enumerable we have $M_\alpha^{\omega} \neq 0$.

In continuing this process we form the following sequence of assemblages:

$M^{\omega}, M_1^{\omega}, M_2^{\omega}, \dots, M_\alpha^{\omega}, \dots$
 each of which is perfect and a

sub-assembly of the preceding assembly. But there can be only an enumerable set of such assemblies for Schoenflies (see Puthoff's paper - Keiten p. 80) has shown that:

"If out of a closed assembly Q , a series of closed sets $Q_1, Q_2, Q_3, \dots$ is formed such that each is a sub-assembly of the preceding assembly and if one arrives finally at a closed assembly T , then this operation must lead to the set T after an enumerable number of steps." By this theorem the subscripts in the above series, namely,

1, 2, 3, $\dots$, $w, w+1, \dots, \infty, \dots$

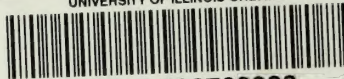
form an enumerable set and therefore must end before it reaches ∞ , the first number of the third class. Therefore $M_2 = 0$ and this is a sufficient condition that $\Phi(x)$ is representable. Hence we may state the following final conclusion:-

6.
If $Q(x)$ is at most pointwise
discontinuous with respect to every
perfect assemblage within the
interval (α, β) , then it is re-
presentable by an infinite series
of continuous functions.





UNIVERSITY OF ILLINOIS-URBANA



3 0112 086762983